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DIGITAL VIDEO: multiple sub–Nyquist coding

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Summary

This Report considers the effect of multiple digital coding of signals, with particular emphasis on sub-Nyquist sampling. Theory is developed which shows that conditions exist whereby no extra degradation due to re-sampling occurs. Experimental work confirming these results is described, and the theory is extended to the use of mixed analogue and digital transmissions, including allowance for practical links.

Issued under the authority of

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DIGITAL VIDEO: MULTIPLE SUB-NYQUIST CODING

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1. Introduction

The use of digital transmission to convey video signals has been the subject of many studies, and it is likely that it will eventually come into service. However, analogue transmission will not be displaced overnight and the two methods will co-exist for some time. During this period a practical link between two points may be partly analogue and partly digital, with perhaps more than one section using each type of transmission.

Thus in designing a system in which an analogue signal is translated into digital form and subsequently decoded to analogue form again we must consider the degree of impairment produced when the process is repeated. For example, a criterion which has often been used[1] is to require that acceptable signal quality is maintained after four successive coding and decoding operations have been performed.

A novel technique for coding composite PAL video signals has been described;[2,3,4] it uses a sub-Nyquist sampling frequency corresponding to twice the PAL colour subcarrier frequency.* The method requires the use of comb filters which introduce some degradation of picture quality, principally a slight loss of high-frequency diagonal resolution. The overall degradation is small and, when the video samples are described using 8-bit pulse-code modulation (p.c.m.), the resulting picture quality is still acceptable for broadcasting purposes.** The method is attractive since the reduction of sampling frequency compared with previous systems (often using $3f_{sc}$ sampling) gives a corresponding reduction in bit rate, offering the prospect of cheaper transmission.

We must now consider what happens when this coding process is repeated. If the effects of the comb filtering stages were cumulative, then the picture quality offered by the system after four successive coding and decoding operations would clearly be unacceptable for broadcasting purposes. The purpose of the work to be described was to investigate the problem both in theory and in practice.

We hope to demonstrate that repeated sub-Nyquist sampling, of the type under consideration,*** does not in fact introduce any fundamental degradation beyond that introduced by the first sampling operation. The coding system does, of course, suffer the cumulative effects (principally quantising noise) of repeated analogue-to-digital-to-analogue conversion, in common with any other digital coding system. Experimental work will also be described, intended to confirm the theoretical results.

2. The sub-Nyquist sampling technique for PAL signals

The sub-Nyquist sampling technique applied to PAL colour television signals is fully described elsewhere,[2,3,4] and only a brief description will be given here.

The working of the system relies on the fact that the spectrum of the video signal has gaps wherein little or no information is normally present. If we consider a monochrome signal initially, most of the energy of this video signal will be clustered around harmonics of the line scanning frequency. If it is sampled at a rate f_o less than twice the highest frequency f_m present, (i.e. sub-Nyquist), alias components will be generated which will overlap the higher frequency part of the signal, (Fig. 1). With an arbitrary choice of signal it would not be possible to distinguish the wanted and alias components but with a video signal we can take advantage of the bunched nature of its spectrum.

Suppose that a sampling rate f_o is chosen which is equal to (n + ½) times line scanning frequency, where n is an integer. Line frequency harmonics in the input signal will now cause the generation after sampling of alias components half-way between line frequency harmonics. Thus with most of the input signal clustered around line frequency harmonics, most of the alias components will be

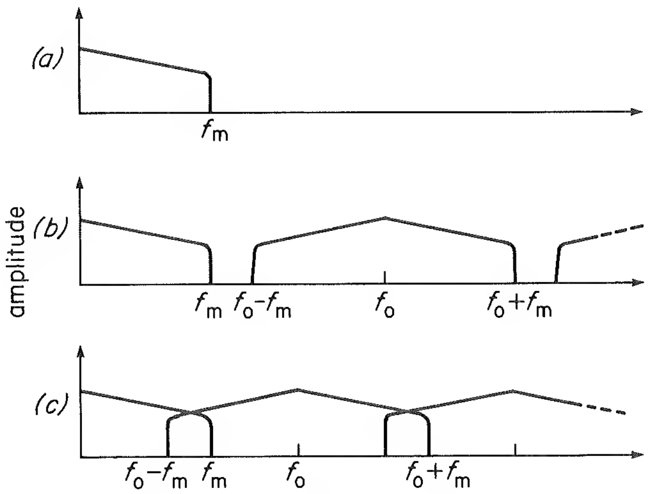


Fig. 1 - Spectra of video signal before and after sampling
(*a*) Spectrum of video signal before sampling
(*b*) Spectrum of sampled video signal for $f_o > 2f_m$
(*c*) Spectrum of sampled video signal for $f_o < 2f_m$

* For System I PAL, colour subcarrier frequency = f_{sc} = 4·43361875 MHz.

** Subjective tests described in Ref. 3 give a grading of 1·5 on the EBU six-point impairment scale.

*** Much of the analysis is of general application: under certain conditions it applies to other types of sub-Nyquist sampling and also to super-Nyquist sampling.

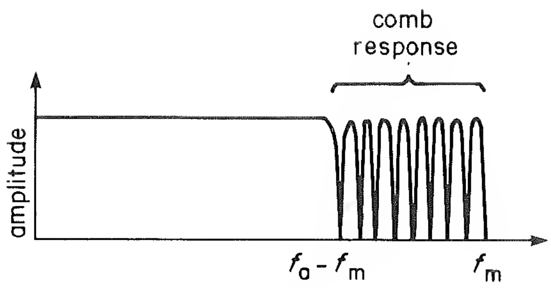


Fig. 2 - Overall response of comb filter

between line frequency harmonics. These may be removed by a comb filter having nulls at (n + ½) times line scanning frequency over the range of overlap (Fig. 2). We have already assumed that little energy is present at these intermediate frequencies in the original signal, so little information is lost. If a similar comb filter is also used to pre-filter the signal before sampling, then those parts of the input signal which would cause aliasing at line harmonics at the output (which the output filter cannot remove) will be removed at the price of a further slight loss of input information.

The degradation caused by the system is a slight loss of high-frequency diagonal resolution.

It may be shown that PAL colour video signals may also be handled by this technique[2,3,4] provided that:

(i) the sampling frequency is exactly $2f_{sc}$ (which is very close to (n + ½) times line scanning frequency),

and (ii) sampling is performed in the correct phase relative to the colour subcarrier.

Additionally, certain constraints are placed on the form of comb filter employed.

In respect of luminance information the behaviour of the system is as outlined above for monochrome signals. It may be shown that with the above provisions the PAL colour subcarrier signal is also conveyed without serious distortion except for the introduction of some loss of resolution of horizontal colour boundaries. The total degradation from a single coding and decoding operation has been assessed subjectively, using a system with 8-bit linear p.c.m. coding, as grade 1·5 on the EBU 6-point impairment scale;[3] this would correspond to roughly grade 4·5 on the new CCIR 5-point impairment scale (CCIR Recommendation 500, Geneva 1974). This performance is probably acceptable for a broadcasting signal transmission circuit.

The block diagram of the basic system is shown in Fig. 3; for clarity the means of generating correctly-phased sampling pulses is omitted.

3. Multiple sub-Nyquist coding

3.1. General considerations

The introduction of digital transmission of video signals is likely to take place on a piecemeal basis. As mentioned in the Introduction, any form of digital coding used must therefore be capable of giving acceptable quality when repeated several times in succession, a useful criterion being the quality obtained after four successive coding operations.

If the sub-Nyquist system described in the previous Section is to be considered for use in digital transmission the effect of applying it repeatedly must be known. In particular, the question arises whether the loss in diagonal resolution, etc., will build up with successive coding operations. This problem is considered in Sections 3.2 and 3.3, and, by an alternative approach, in Section 3.4. It will be shown that the degradations caused by comb filtering and sub-Nyquist sampling are not in theory cumulative.

A further consideration is the effect of interposing a practical analogue link and practical filters between successive coding operations. This will be discussed in Section 3.5.

Finally, Section 3.6 will deal with the effects of repeated quantisation.

3.2. Analysis of repeated sampling

In this Section we shall consider the effects of repeated sampling of the signal. The process of digital coding does, of course, involve both sampling and quantising, but the latter will not be considered for the present.

Fig. 4 shows two systems which will be analysed. Fig. 4(a) represents the basic sub-Nyquist system consisting of a pre-coding comb filter, sampler and post-coding comb filter. The comb filter responses are $X_1(f)$ and $X_2(f)$ respectively. Fig. 4(b) shows a system where sampling has been repeated, but with some filter of arbitrary frequency response $G(f)$ placed between the samplers. For example,

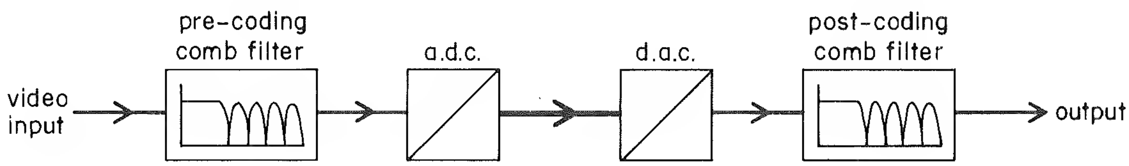
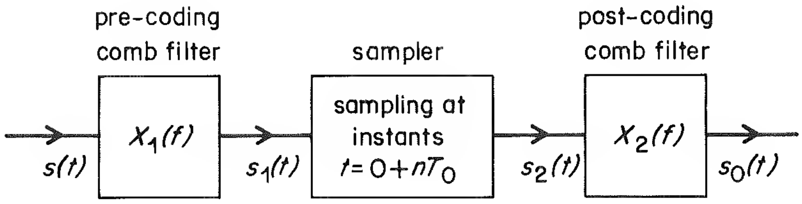
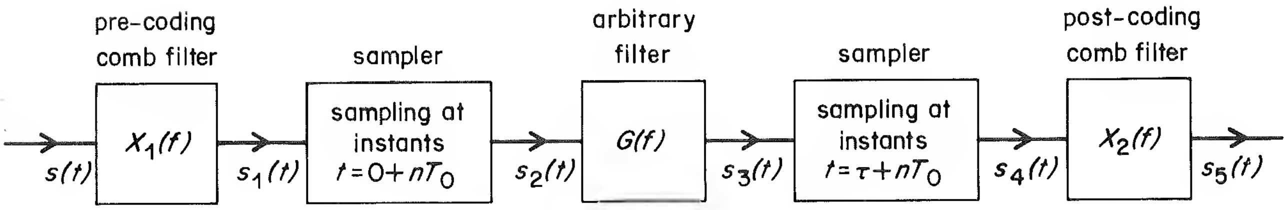


Fig. 3 - Block diagram of sub-Nyquist coding system. The a.d.c. samples at $2f_{sc}$



(*a*)



(*b*)

Fig. 4 - The systems for analysis
(*a*) Single-sampling system (*b*) Repeated-sampling system

in a simple repetition of the basic system, $G(f)$ would consist of a post-coding comb filter and pre-coding comb filter in cascade. By comparing the two systems we may determine the extra degradations, if any, introduced by the second sampling process.

Certain simplifying assumptions are made:

(i) $G(f) = 0$, $|f| \geqslant f_0$ where $f_0 = {}^1/T_0$ is the sampling frequency.

(ii) $S(f) = 0$, $|f| \geqslant f_0$ where $S(f)$ is the spectrum of the input signal $s(t)$, given by the Fourier transform relations:

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt$$

($\omega = 2\pi f$, this notation is used throughout the report).

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j\omega t}df$$

(iii) The samplers are assumed to have normalised gains and have their outputs low-pass filtered so that, for example, $S_2(f) = 0$ for $|f| \geqslant f_0$. This is valid since the samplers are always followed by filters of response $G(f)$ or $X_2(f)$, both of which have this low-pass property. The advantage is that high-order alias components may be neglected at an earlier stage in the analysis.

(iv) $G(f)$ is a real filter so that its impulse response $g(t)$ is real.

Single-sampling system, Fig. 4(*a*)

The filter responses are known, so that

$$S_1(f) = X_1(f)\,S(f) \tag{1}$$

and

$$S_0(f) = X_2(f)\,S_2(f) \tag{2}$$

The sampler samples $s_1(t)$ at a rate f_0; according to classical sampling theory we may write

$$S_2(f) = \sum_{n=-\infty}^{\infty} S_1(f - nf_0)$$

which, with the assumptions made in (iii) above, simplifies to

$$S_2(f) = S_1(f) + S_1(f+f_0) + S_1(f-f_0) \tag{3}$$

This is the result illustrated in Fig. 1.

Repeated-sampling system, Fig. 4(*b*)

We are interested in the additional effect of the second sampling operation, that is to say the difference between signals $s_0(t)$ and $s_5(t)$ as shown in Fig. 4. It will be useful to define an 'effective insertion transfer function', $Y(f)$, of the arbitrary filter, of response $G(f)$, followed by the second sampler.

Let
$$Y(f) = \frac{S_5(f)}{S_0(f)} = \frac{X_2(f)\,S_4(f)}{X_2(f)\,S_2(f)}$$

i.e.
$$Y(f) = \frac{S_4(f)}{S_2(f)} \tag{4}$$

At this stage the mathematics becomes somewhat complicated and, for clarity, further working has been relegated to the Appendix, Section 7. Section 7.1 derives the equation describing the effect of sampling at instants $t = \tau + nT_o$, 7.2 applies this to the repeated-sampling system and 7.3 derives an expression for the function $Y(f)$ we have defined. The result, for positive f in the range $0 \leqslant f < f_o$, is Equation (A6) of Section 7.3:

$$Y(f) = G(f) + e^{-j\omega_o\tau}\, G(f - f_o) \tag{5}$$

For negative f the second term of the R.H.S. must be changed to take account of the fact that $G(f - f_o)$ is then zero (see Equation (A7) of 7.3). An alternative approach is to use the result of 7.3 that $Y(f)$ behaves like a real filter and therefore obeys the relation

$$Y(f) = Y^*(-f)$$

To summarise, if we know the response $G(f)$ of the arbitrary filter and the delay τ between the operations of the two samplers we can specify the effective insertion transfer function $Y(f)$ which describes how the overall response of the system of Fig. 4(*b*) differs from that of the system of Fig. 4(*a*).

3.3. Requirements for no added distortion

We are particularly interested in the condition where the insertion of the combination of $G(f)$ and the second sampler causes no added distortion, so that $s_s(t)$ is identical to $s_o(t)$ except for some delay T. Translated into the frequency domain this implies

$$Y(f) = e^{-j\omega T} \tag{6}$$

Substituting this condition into Equation (5) we obtain the general solution for no added distortion:

$$e^{-j\omega T} = G(f) + e^{-j\omega_o\tau} G(f - f_o) \text{ for } 0 \leqslant f < f_o \tag{7}$$

(The reader may obtain a corresponding equation for negative f using Equations (6) and (A7) — the rest of this section will consider only positive f and rely on the assumption that $g(t)$ is real).

Having obtained this general result it will be useful to consider the more special case where $G(f)$ has uniform group delay T. In other words

$$G(f) = A(f) e^{-j\omega T} \tag{8}$$

where $A(f)$ is real, and denotes the amplitude-frequency response of the filter.

Substituting (8) into (5) we obtain

$$Y(f) = A(f)e^{-j\omega T} + e^{-j\omega_o\tau} A(f - f_o) e^{-j(\omega - \omega_o)T}$$

i.e. $$Y(f) = e^{-j\omega T}\left\{A(f) + e^{-j\omega_o(\tau - T)} A(f - f_o)\right\} \tag{9}$$

We note that in the practical system which we are modelling the second sampler arranges its phase of sampling using the colour subcarrier as a reference so that it samples at the peaks and troughs thereof.* τ is thereby chosen to satisfy

$$\tau = T_c + nT_o$$

where T_c denotes the phase delay introduced by the network $G(f)$ at $f = f_o/2$, the colour subcarrier frequency. If the network has uniform group delay T then obviously $T_c = T$. Thus $\tau = T + nT_o$ so that (9) simplifies to

$$Y(f) = e^{-j\omega T}\left\{A(f) + A(f - f_o)\right\} \tag{10}$$

and we may see at once that $Y(f)$ is a pure delay T provided that

$$A(f) + A(f - f_o) = 1 \tag{11}$$

This is the simple condition for no additional distortion when $G(f)$ has uniform group delay.

Two important conclusions may now be drawn:

(i) It may be shown that the system comprising a post-coding and pre-coding comb filter in cascade has a frequency response which satisfies (11) combined with a uniform group delay.** Therefore, no extra degradation is introduced when the sub-Nyquist sampling arrangement of Fig. 3 is repeated by joining another in cascade. It follows that an indefinite number of such systems may be cascaded without further degradation.

(ii) It is not necessary to use a comb filter pair between sub-Nyquist sampling operations. Any filter with uniform group delay whose frequency response $A(f)$ satisfies Equation (11) may be used. An endless number of such filters may be envisaged, but a set of particular interest comprises those with smooth low-pass responses of which an example is shown in Fig. 5. The response is antisymmetric about $f = f_o/2 (= f_{sc})$, at which point it has the value ½.

This suggests a method of cascading digital and analogue links whereby an intermediate analogue link would carry a non-standard video signal. Fig. 6(*a*) shows the possible arrangement. The performance would be identical to that given by the conventional sub-Nyquist arrangement (Fig. 6(*b*)) but replaces two comb filters by one antisymmetric low-pass filter. Assuming 5·5 MHz bandwidth for analogue equipment this would require $A(f)$ of the filter to be zero for $f \geqslant 5{\cdot}5$ MHz. This implies $A(f) = 1$ for f from 0 to the 'mirror' frequency of 5·5 MHz, which is 8·87 − 5·5 = 3·37 MHz. Between 3·37 MHz and 5·5 MHz the response is a smooth roll-off antisymmetric about 4·43 MHz, at which frequency $A(f) = ½$. The signal handled by the analogue link would be non-standard in that the colour

* This is the same as is performed by the first sampler which handles a signal whose subcarrier phase has been made uniform by the pre-coding comb filter (see Ref. 3).

** The analysis for a pair of simple comb filters (combing over the whole band) is given in the Appendix, Section 7.4. The analysis for the more complex filters actually used is more protracted.

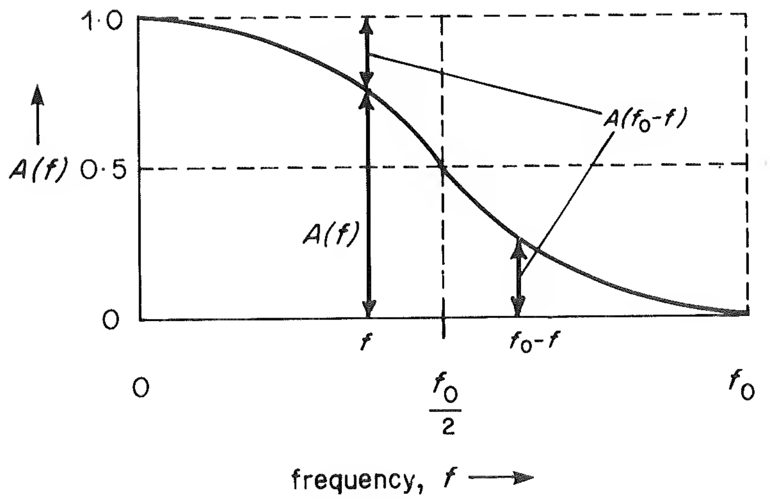
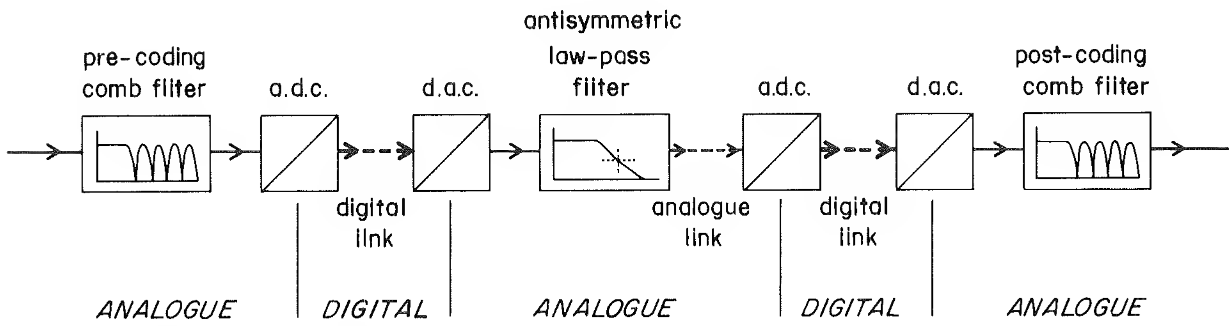
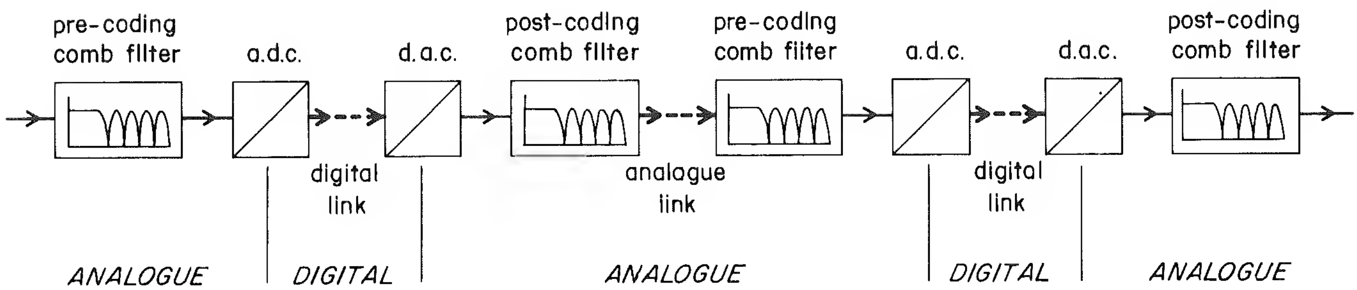


Fig. 5 - Antisymmetric low-pass filter response



(a)



(b)

Fig. 6 - Cascaded digital and analogue links

(*a*) Using a low-pass filter in the intermediate stage (*b*) Using comb filters in the intermediate stage

information would be of uniform phase: this is the same as the output of the original pre-coding comb filter. It is interesting to note that this technique could be used to transmit a PAL video signal in a bandwidth of less than 5·5 MHz (see Section 3.5).

The possible advantage of such a system is that the low-pass filter of the type specified might be cheaper to build than a pair of comb filters; against this must be weighed the disadvantage that the signal carried on the analogue link is non-standard, possibly presenting difficulties of monitoring.

A slight modification may be proposed whereby low-pass filters are placed before and after the intermediate analogue link. In this case the combined response of the two filters should have the form prescribed for $A(f)$ above. This system would reduce slightly the effects of noise added by the analogue link.

In conclusion, we have shown that it is possible in principle to repeat sub-Nyquist sampling without any added degradation, provided certain conditions are met. Moreover, the equations derived may be used to calculate the degradations introduced when the performance of a practical system departs from the ideal conditions. This point will be pursued in Section 3.5.

3.4. An alternative approach to the conditions for distortionless re-sampling

The analysis presented so far has used the frequency domain approach. This is easy to use since the effects of cascaded linear processes may be compounded by simple multiplication of their frequency responses. However, the meaning of the results is not easy to visualise. In this Section we will examine the problem from the viewpoint of the time domain.

Consider the system of Fig. 4(*b*) once more. The first sampler produces an output, $s_2(t)$, which may be thought of as a series of impulses whose amplitudes represent the sampled values of the input signal $s_1(t)$, i.e.

$$s_2(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - nT_0)$$

where $\delta(t - \tau)$ denotes a unit impulse at $t = \tau$ and $a_n = s_1(nT_0)$. (12)

This is of course an example of pulse-amplitude modulation (p.a.m.). The train of impulses passes through the arbitrary filter before resampling. No distortion is introduced if the original set of impulses is reproduced again by the second sampler, i.e. we require

$$s_4(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - nT_0 - T), \qquad (13)$$

which is $s_2(t)$ delayed by T

and where clearly $\qquad T = \tau + kT_0, \qquad (14)$

k being some fixed integer.

Consider the very simple case when $a_0 = 1$ and all other a_n's are zero, i.e.

$s_2(t) = \delta(t)$, a unit impulse at the time origin.

The output of the filter is therefore its impulse response, $g(t)$, thus $s_3(t) = g(t)$. This signal is sampled at instants $t = nT_0 + \tau$ to produce an output which we require to be simply the original unit impulse delayed by T, i.e.

$$\delta(t - T) = g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_0 - \tau) \qquad (15)$$

Thus we have a constraint on the values of $g(t)$ at instants $t = \tau + nT_0$, namely $g(T) = 1$

and $g(T + mT_0) = 0$ for all values of m except $m = 0$.

The impulse response $g(t)$ is constrained to contain a series of zeros at all sampling instants except the displaced time origin T. This is the condition for no distortion of a single impulse, which has merely undergone a delay T. Any real signal with an arbitrary sequence of a_n's will similarly have each of its impulses simply delayed and this is also undistorted provided the constraint on $g(t)$ is satisfied.

We may translate (15) into the frequency domain. Noting that the R.H.S. represents sampling $g(t)$ at $t = \tau + nT_0$ we may use the result of Section 7.1 for its spectrum. Thus, taking the Fourier transform of both sides we obtain

$$e^{-j\omega T} = \sum_{n=-\infty}^{\infty} G(f - nf_0) e^{-jn\omega_0 \tau} \qquad (16)$$

If we apply our assumption that $G(f) = 0$ for $|f| \geqslant f_0$ we obtain for the frequency range $0 \leqslant f < f_0$

$$e^{-j\omega T} = G(f) + e^{-j\omega_0 \tau} G(f - f_0)$$

This is identical to Equation (7) derived in Section 3.3. As before, the equation for negative f is also easily obtained.

Thus by argument in the time domain we have obtained the same condition for distortionless re-sampling as obtained earlier by a frequency domain approach. In so doing we have uncovered the meaning of this condition in the time domain, namely that the impulse response of the filter, $g(t)$, shall be zero at all sampling points except the displaced time origin T. Each impulse resulting from the first sampling operation is blurred out in time by the filter, but because of these zeros in $g(t)$ there is no crosstalk from one sample to another at the points of sampling. Thus the original sample values are recovered by the second sampler.

In fact our result is already well known as the condition for zero intersymbol interference in telegraph signalling, first described by Nyquist in 1928.[5] We may think of the series of impulses, $s_2(t)$, leaving the first sampler as being a series of telegraph symbols. The filter $G(f)$ then represents the telegraph transmission line which delivers a distorted version of the series of symbols. However, by sampling at the appropriate instants the correct symbols may be decoded provided that there is no intersymbol interference, which requires that the condition we have derived is satisfied.

The analogy with telegraphy is still closer when we consider the actual sub-Nyquist coding system. Each sampler is then followed by a quantiser, the whole constituting an analogue-to-digital converter (a.d.c.) so that each sample is described by, say, J bits. A d.a.c. reproduces each sample which is thus constrained to one of 2^J levels. This now corresponds to 2^J-level telegraph signalling, conveying J bits of information per transmission symbol.

As a final illustration in the time domain we include

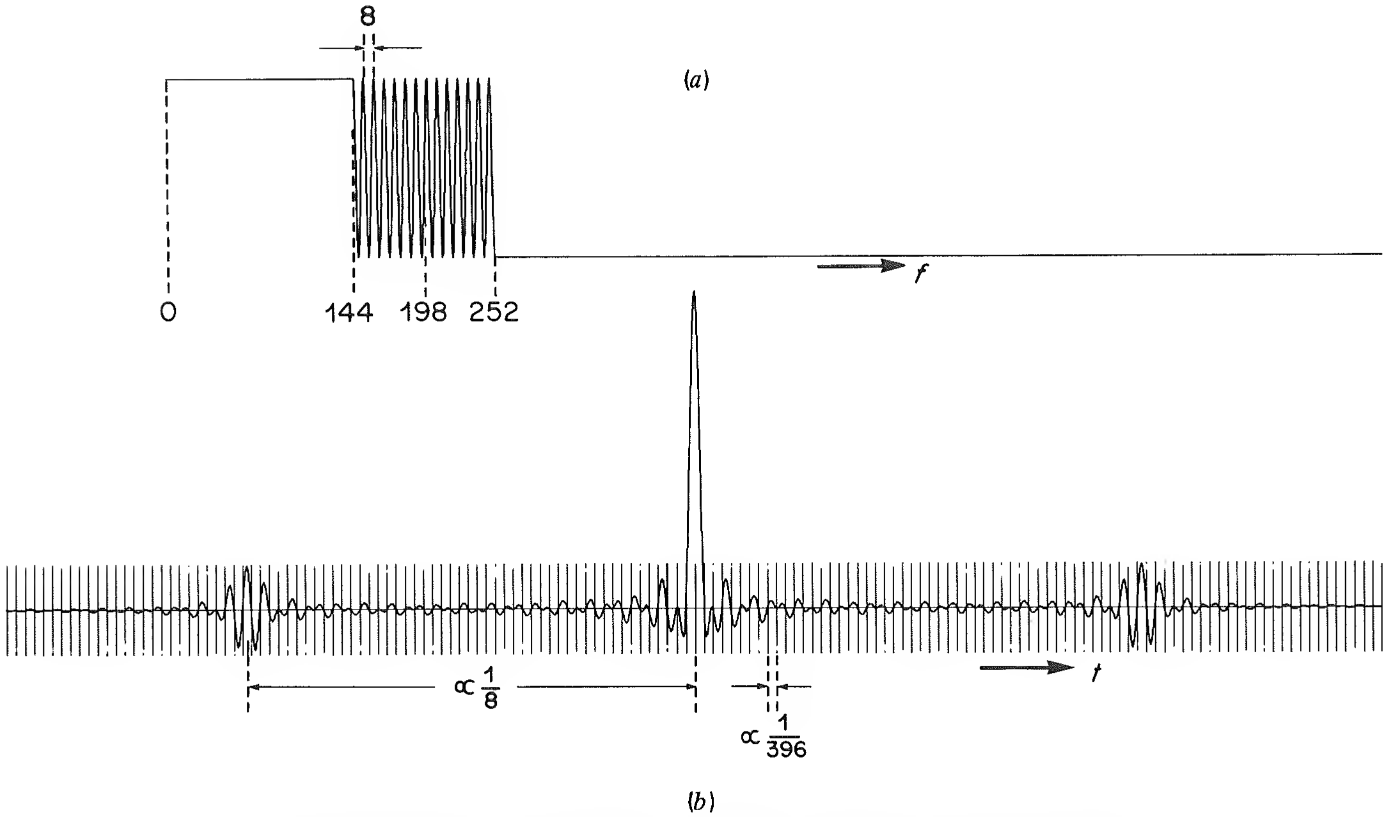


Fig. 7 - Illustration of the properties of the impulse response of a pair of comb filters

(*a*) Assumed frequency response of a pair of comb filters (*b*) The resulting impulse response, showing sampling points

two Figures which bring out the point that the impulse response of an antisymmetric filter of the form under discussion is zero at all sampling points except the one at the (displaced) time origin. The Figures* were produced by means of a computer programme which performs Fourier transformation, the results being drawn by a plotter under computer control.

Fig. 7 shows the frequency response and impulse response of a comb filter pair. The comb spacing chosen is coarser than for the filters actually used in order to give a clearer picture, but the important point is that the sampling frequency (396 arbitrary units) is still (n + ½) times the comb spacing (8 units). 198 units corresponds to $f_0/2$; note that the frequency response is antisymmetric about this point, at which it has the value ½. The sampling instants are superimposed on the impulse response and it may be seen that the value of the impulse response is zero at every sampling instant except the origin. This shows why resolution is not progressively lost as sub-Nyquist sampling systems using comb filters are cascaded. Each comb filter does average h.f. information from two successive scanning lines, but which the resulting signal is sampled this averaged information is discarded.

Fig. 8 shows the frequency response and impulse response for a typical smooth antisymmetric low-pass response. In this case the impulse response extends over a shorter time span but retains the property that it is zero at every sampling instant except the origin.

* The authors are indebted to A.H. Jones and R.E. Davies for these Figures.

3.5. The effect of interposing a practical link

In this Section we shall consider the effect of given imperfections in an intermediate analogue link when mixed digital and analogue transmission, of the form of Fig. 6(*a*) or (*b*), is in use. The main purpose is to see whether the performance specification of the analogue link needs to be any more stringent than that normally applied for analogue transmission.

Start from Equation (5) which gives the 'effective insertion transfer function', $Y(f)$:

$$Y(f) = G(f) + e^{-j\omega_0\tau}\, G(f - f_0)$$

where $G(f)$ is the combined response of the link and filters. Write $G(f) = G_1(f)G_2(f)$, where $G_1(f)$ is the response of the filters and $G_2(f)$ is the response of the link. Equation (5) thus becomes

$$Y(f) = G_1(f)G_2(f) + e^{-j\omega_0\tau}G_1(f - f_0)G_2(f - f_0) \qquad (17)$$

Suppose that the filters have an antisymmetric amplitude response $A_1(f)$ with uniform group delay T_1 so that the condition for distortionless re-sampling (Section 3.3) would be satisfied if the link were perfect.* Specify that $A_1(f) = 0$ for $f \geqslant 5{\cdot}5$ MHz so that the link has to handle a signal of normal video bandwidth. This implies that $A_1(f)$ is unity from 0 to 3·37 MHz.

* We have shown that this is true for either a pair of comb filters or the low-pass filter of Fig. 5.

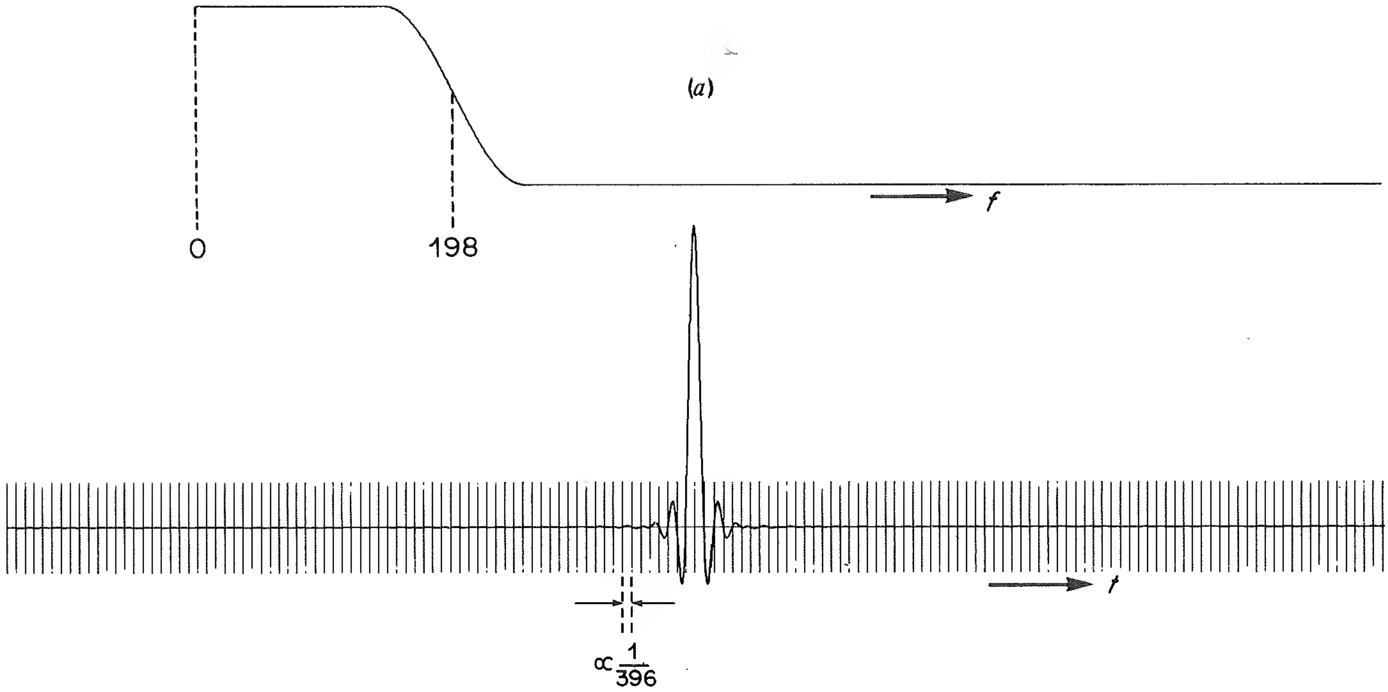


Fig. 8 - Illustration of the properties of the impulse response of a low-pass filter with frequency response antisymmetric about $f_o/2$ (= 198 units in this case)

(a) Assumed low-pass frequency response (b) The resulting impulse response, showing sampling points

For this range Equation (17) then becomes

$$Y(f) = 1.G_2(f)$$

so that the insertion response in the band 0 to 3·37 MHz equals the response of the link exactly in both amplitude and phase.

We now pass to the h.f. response. To keep the analysis simple we shall assume that the link suffers gain-frequency variation or group delay variation but not both.

Case (i) Link has uniform group delay T_2 but varying amplitude response $A_2(f)$.

Substituting in (17) we obtain

$$Y(f) = \left[A_1(f).A_2(f) + \left\{1 - A_1(f)\right\}\left\{A_2(f - f_o)\right\}\right] e^{-j\omega(T_1 + T_2)} \qquad (18)$$

provided that τ, the sampling delay, is chosen to match the total group delay $(T_1 + T_2)$. We are concerned with the sensitivity to departures of $A_2(f)$ from the ideal value of unity. If anything this sensitivity may be reduced compared with simple analogue transmission since the amplitude response $|Y(f)|$ at any frequency f between 3·37 MHz and 5·5 MHz is a weighted average of the link response at frequencies f and $(f - f_o)$, the weighting depending on the form of $A_1(f)$. For example, suppose a ±1 dB tolerance applies to $A_2(f)$. The maximum variation in $|Y(f)|$ is still ±1 dB but if the errors at f and $(f - f_o)$ are of opposite sign the variation in $|Y(f)|$ will be reduced to less than ±1 dB.

An interesting special case is when $A_1(f)$ has the smooth low-pass response of Fig. 5. As the cut-off about $f_o/2$ (subcarrier frequency) is sharpened, the response of the link in the range above $f_o/2$ becomes less and less important. In the limit, $A_1(f)$ becomes a perfect 4·43 MHz low-pass filter and the link performance above 4·43 MHz is quite unimportant. This result may appear somewhat startling until it is remembered that the ideal Nyquist signalling rate with 4·43 MHz bandwidth, i.e. $f_o/2$, is f_o symbols/second. The first sub-Nyquist sampling operation, sampling at f_o, has of course reduced the information content of the video signal to this rate. It can therefore be transmitted subsequently within a bandwidth $f_o/2$ without further degradation if ideal filters are used.

Case (ii) Link has uniform amplitude response but varying group delay

We write $G_2(f) = e^{+j\phi(f)}$ where $\phi(f)$ = phase response of link.

$$\text{Let } \phi(f) = \phi'(f) - \omega T_c \text{ where } T_c = \frac{-\phi(f_o/2)}{\omega_o/2}$$

$$\text{thus } \phi'(f_o/2) = 0.$$

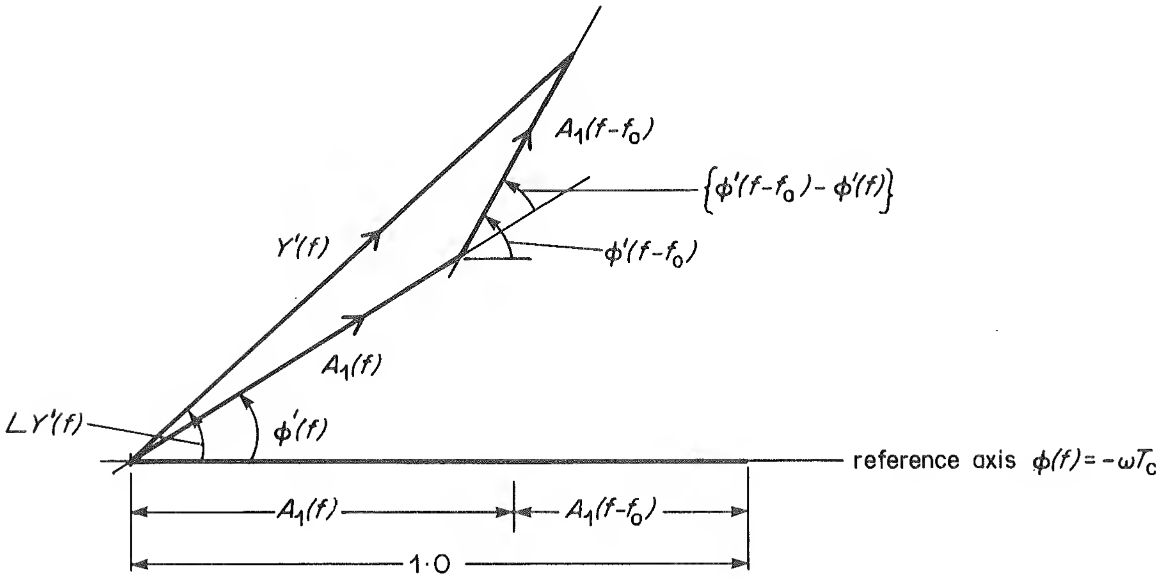


Fig. 9 - Use of vector diagram to determine the overall response when a link suffers from group-delay variation (Section 3.5)

Substituting, $Y(f) = A_1(f)e^{-j\omega T_c}e^{j\phi'(f)} + e^{-j\omega_0\tau}(1 - A_1(f))e^{-j(\omega-\omega_0)T_c}e^{j\phi'(f-f_0)}$

We note that the second sampler uses the subcarrier as a phase reference so that $\tau = T_c$. The equation thus simplifies to

$$Y(f) = e^{-j\omega T_c}\left\{A_1(f)e^{j\phi'(f)} + (1 - A_1(f))e^{j\phi'(f-f_0)}\right\} \quad (19)$$

The term within the brackets $\{\ \}$, say $Y'(f)$, describes the deviation of $Y(f)$ from unity gain and uniform group delay equal to T_c. This is illustrated by the vector diagram of Fig. 9.

The first observation to make is that the angle $\angle Y'(f)$ (representing the departure of the phase response from uniform group delay) is less than or equal to the greater of $\phi'(f)$ or $\phi'(f-f_0)$. This means that the phase response is no worse than that of the link.

The amplitude response is represented by the length of the vector $Y'(f)$. At any given frequency this is reduced from the ideal value of 1·0 as the angle $\{\phi'(f-f_0) - \phi'(f)\}$ increases. Thus the *phase* response of the link, $\phi(f)$, has given rise to a variation of the output in *amplitude*, as well as phase. To see whether this is within acceptable limits we need to know the likely variation of $\{\phi'(f-f_0) - \phi'(f)\} = \theta(f)$, say.

Link specifications are frequently written in terms of group delay variation rather than phase response. Appendix 7.6 shows how $\theta(f)$ may be determined from areas under a group-delay curve. Fig. 10 is a group-delay template for a hypothetical reference circuit taken from the CCIR Green Book, Geneva, 1975, Vol. XII, p. 94. On it have been superimposed two curves which represent those characteristics giving rise to the greatest value of $\theta(f)$. (The response from 0 to 3·37 MHz is immaterial for this purpose.) With either of these curves the calculated value of $\theta(f)$ for f = 3·37 or 5·5 MHz is 85°, using Equation (A11) of the Appendix. The minimum value of $|Y(f)|$ is then given when $A_1(f) = A_1(f-f_0) = ½$; $|Y(f)|$ is then 0·73, i.e. −2·65 dB relative to 1·0. Suppose $A_1(f)$ takes the form of two comb filters in cascade. The minimum value of $|Y(f)|$ will then occur at frequencies where $A_1(f) = ½$, namely half way between peaks and troughs. These correspond to high frequency diagonal luminance information. The effect will only be significant towards the extremes of the combed range (3·37 and 5·5 MHz) because $\theta \to 0$ as $f \to f_0/2$. In contrast, if $A_1(f)$ is a smooth antisymmetric low-pass filter response the amplitude reduction we have described will be minimised since then $\theta \to 0$ as $A(f) \to ½$.

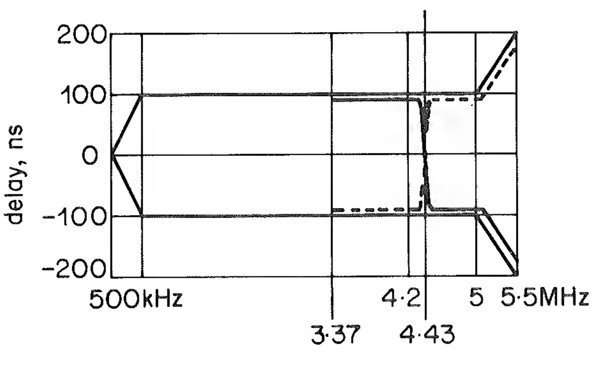


Fig. 10 - CCIR group delay template showing hypothetical worst responses as regards multiple coding

(- - - - - - represent those group delay characteristics which give the greatest magnitude of $\theta(f)$)

Thus, if a practical link had a group-delay characteristic of the extreme form shown in Fig. 10 there would be a possible advantage in using the low-pass characteristic for $A_1(f)$ in preference to the comb filter pair.

We have considered amplitude and phase response variations separately; to consider both would require a

slight extension of the vector diagram technique illustrated in Fig. 9.

In conclusion, we have shown that the introduction of an intermediate *imperfect* analogue link to a PAL sub-Nyquist system causes no drastic ill-effects. The insertion transfer function $Y(f)$ resulting from such a link and the subsequent sub-Nyquist system obeys the following:

(i) $Y(f)$ is equal to $G_2(f)$ (the link response) at frequencies below 3·37 MHz.

(ii) For frequencies above 3·37 MHz $Y(f)$ depends on the response of the link at f and also at the 'mirror' frequency $(f - f_0)$, the two contributions being weighted according to the filter response $A_1(f)$ chosen.

(iii) For a link with varying amplitude response but uniform group delay the maximum departure of $Y(f)$ from unity will be no worse than the response of the link and may be rather better if $A_1(f)$ is carefully chosen. For example, if a link has poor h.f. response above subcarrier frequency then a low-pass filter with antisymmetric response cutting off fairly sharply above subcarrier frequency should be used. $Y(f)$ will then show a better h.f. response than the link by itself.

(iv) If a link has non-uniform group delay then $Y(f)$ will show group delay variation and also amplitude response variation. When comb filters are used with such a link a small loss of diagonal information may occur but, for links satisfying the template of Fig. 10, this loss will be less than would be given by the comb filter without re-sampling. On the other hand, a low-pass filter will introduce a ripple in the h.f. response so that the information loss is smaller but more general.

3.6. Multiple coding including quantising

The discussion so far has considered repeated *sampling* operations in cascade. We shall now briefly consider the *quantising* that accompanies each sampling operation in a practical system of cascaded digital codecs.

The assumption usually made is that when codecs are cascaded each contributes noise of mean-squared value equal to $\delta^2/12$, where δ is the quantum spacing; 'noise-power addition' is thus assumed. As doubling δ corresponds to using one less bit per sample we obtain the often-quoted result that using four codecs in cascade is equivalent to the loss of one bit per sample.

This is not strictly valid. It rests on the assumption of a uniform probability density of the amplitude distribution of the signal which is quantised.[6] This is perhaps reasonably true for the input to the first codec, but the signal applied to the succeeding stages may not necessarily have this property, especially if the sampling is in some way locked to the signal.

Consider a special case in which:

(i) the quantisers are identical and produce output levels V_i for inputs in the ranges $V_i - \delta/2$ to $V_i + \delta/2$ (where levels V_i are spaced by δ);

(ii) any noise introduced between the quantisers is very much smaller than δ;

(iii) sampling is in the same phase w.r.t. the signal in each case;

(iv) the codecs are connected by a circuit of unity gain; and

(v) an antisymmetric filter with uniform group delay is placed between the output of each quantiser and the input to the next so that Equation (11) of Section 3.3 is satisfied. It is easy to see that the output of each sampler (excluding the first) will be the same as the output of the previous one. Thus no further quantising noise is introduced after the first sampling and quantising operation.

In practice condition (i) could be relaxed somewhat to: each quantiser produces outputs V_i' for inputs in the ranges $V_i - \delta/2$ to $V_i + \delta/2$ where $V_i - \delta/2 < V_i' < V_i + \delta/2$, provided that the added noise (condition (ii)) is reduced indefinitely as the V_i' approach the extremes of the tolerance specified. The quantising noise in this case is correlated; noise powers do not add and although the worst-case mean-squared noise is as much as $\delta^2/4$ after only two codings, this remains the worst-case figure for an indefinite number of codings. This condition is more nearly practicable: it would be satisfied by a perfect a.d.c. with a d.a.c. having a tolerance of $\pm\delta/2$, or by an a.d.c. and d.a.c., each with $\pm\delta/4$ tolerance.

To analyse any further general relaxation of the conditions becomes very difficult, but it is possible to consider some other special cases, for example the subset of quantising noise known as contouring. This results when a slowly-varying signal is quantised; the output of the quantiser jumps from one quantum level to the next, spending a long period on each.

Suppose that a sawtooth waveform is quantised. The result will be a staircase waveform with steps of height δ and having a regular spacing. If this signal is then re-quantised under the relaxed conditions given above the resulting contours will be irregularly spaced (in amplitude and position), but equal in number to those of the first quantising operation. If the gains are not accurately matched so that the quantum spacing of the second operation, δ', is not equal to that of the first, δ, then a 'beating' effect takes place, illustrated in Fig. 11. Contouring is made more visible, but note that the figure shows an extreme case for clarity, in which the gain mismatch is large.

4. Experimental work

4.1. Introduction

It was considered worthwhile to confirm the theoretical results by a practical experiment in which a video

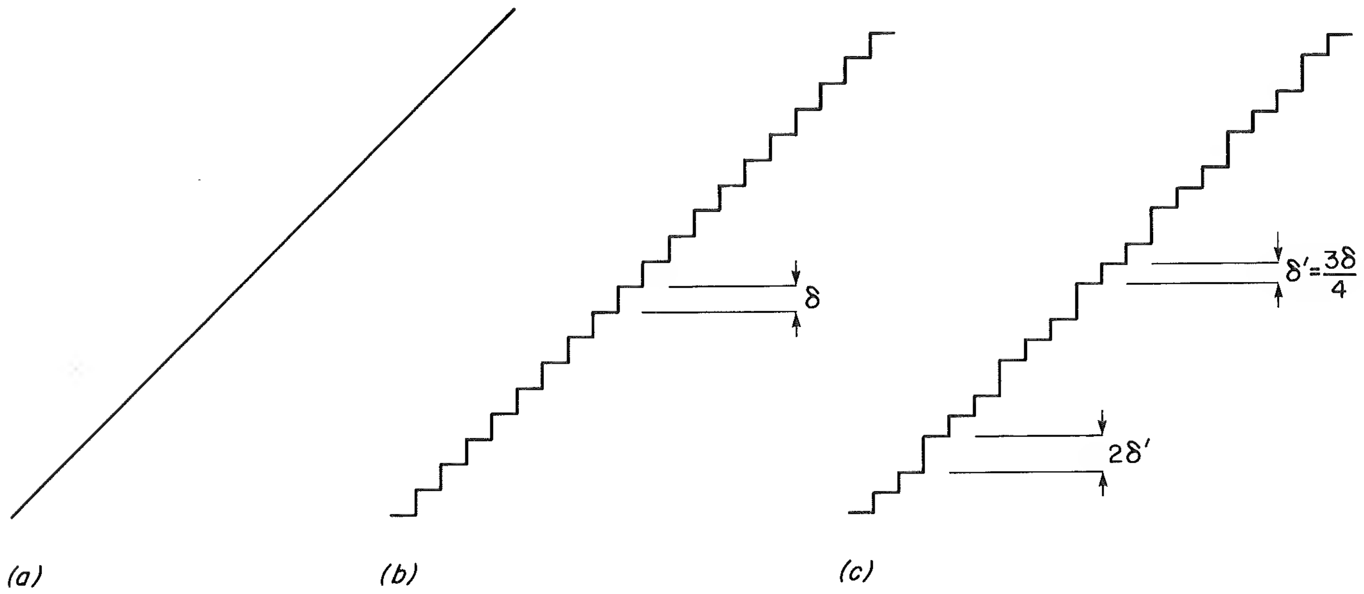


Fig. 11 - The effects of multiple coding on contouring when there is a gain mismatch
(*a*) Ramp waveform (*b*) The result after sampling and quantising (*a*) with quantum step δ
(*c*) The result after sampling and quantising (*b*) with quantum step $\delta' = 3\delta/4$

signal would be subjected to sub-Nyquist coding and decoding several times in succession to see if picture quality would deteriorate further after the first operation The suggestion of using an antisymmetric low-pass filter as an alternative to the comb filter pair could also be tried.

How was this to be done? A succession of sub-Nyquist codecs actually connected in cascade was impracticable as this volume of equipment was not to hand. A system of recording and successive dubbing using only one codec was thus needed. However, an analogue quadruplex video tape recorder (VTR) could not be used because:

(i) the defects of analogue tape recording would build up with successive dubs, masking the effects we wished to observe;

and (ii) two analogue VTRs would be needed in the intermediate stages, one recording and the other playing back. Only one machine was available.

The solution lay in the use of an experimental digital recorder designed and built in the BBC Research Department.[7]

4.2. The experimental arrangement incorporating the digital recorder

The experimental digital recorder can record PAL video signals on magnetic tape in digital form. As organised at the time of the experiments it used the sub-Nyquist $2f_{sc}$ sampling system, with 7-bit p.c.m. coding. Facilities are provided for error correction and concealment. It has separate record and play heads and associated circuits, so that in normal use play-after-record off-tape monitoring is provided after the fashion of normal studio audio recorders. This is a facility which is not provided by analogue VTR machines.

When it is desired to examine the results of multiple processing the direction of tape motion is reversed so that the play head precedes the record head. Suppose the tape has an rth generation of processing recorded on it. This may be replayed, processed once more and recorded 'downstream' of the play head as an $(r + 1)$th generation recording. Thus an nth generation recording is produced by:

(i) first recording from the picture source via the processing,

and (ii) performing the dubbing and re-processing operation $(n - 1)$ times.

The method is easily applied to our experiment since the same clock rate $(2f_{sc})$ is used by the experimental processing and the recorder. Fig. 12 shows the arrangement. The filter shown as $G(f)$ was in fact either:

(i) a post-coding comb filter and a pre-coding comb filter in cascade,

or (ii) a low-pass filter of approximately antisymmetric frequency response.

The analogue VTR is not essential and was only used for convenience so that the results might be studied later without the digital recorder. Note that the final replay (stage three of the figure) uses no additional equipment to that

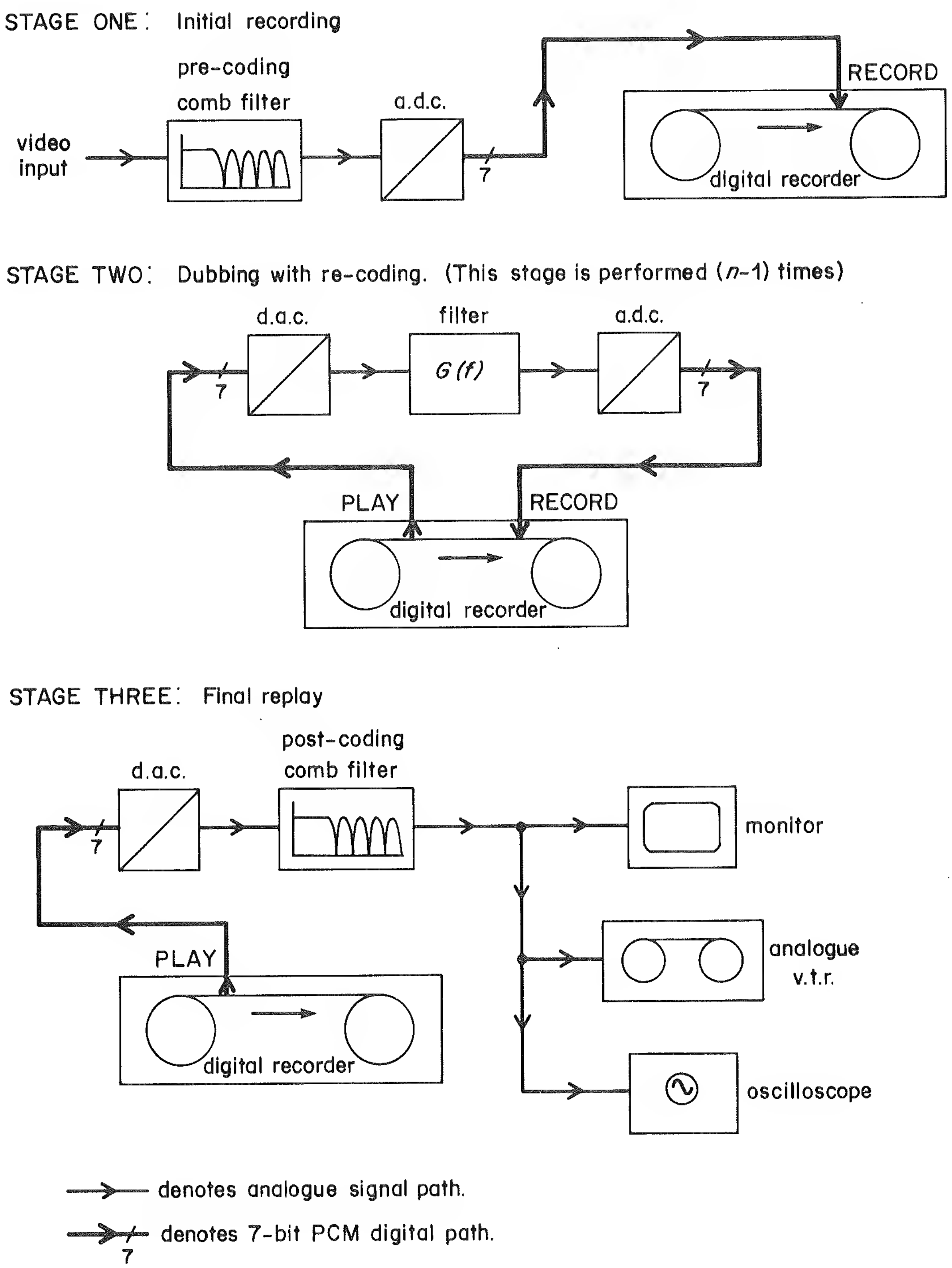


Fig. 12 - Method of producing an nth generation recording

normally required for playback using the digital recorder, so the 'master' digital tapes may be viewed at any later time when the digital recorder is available.

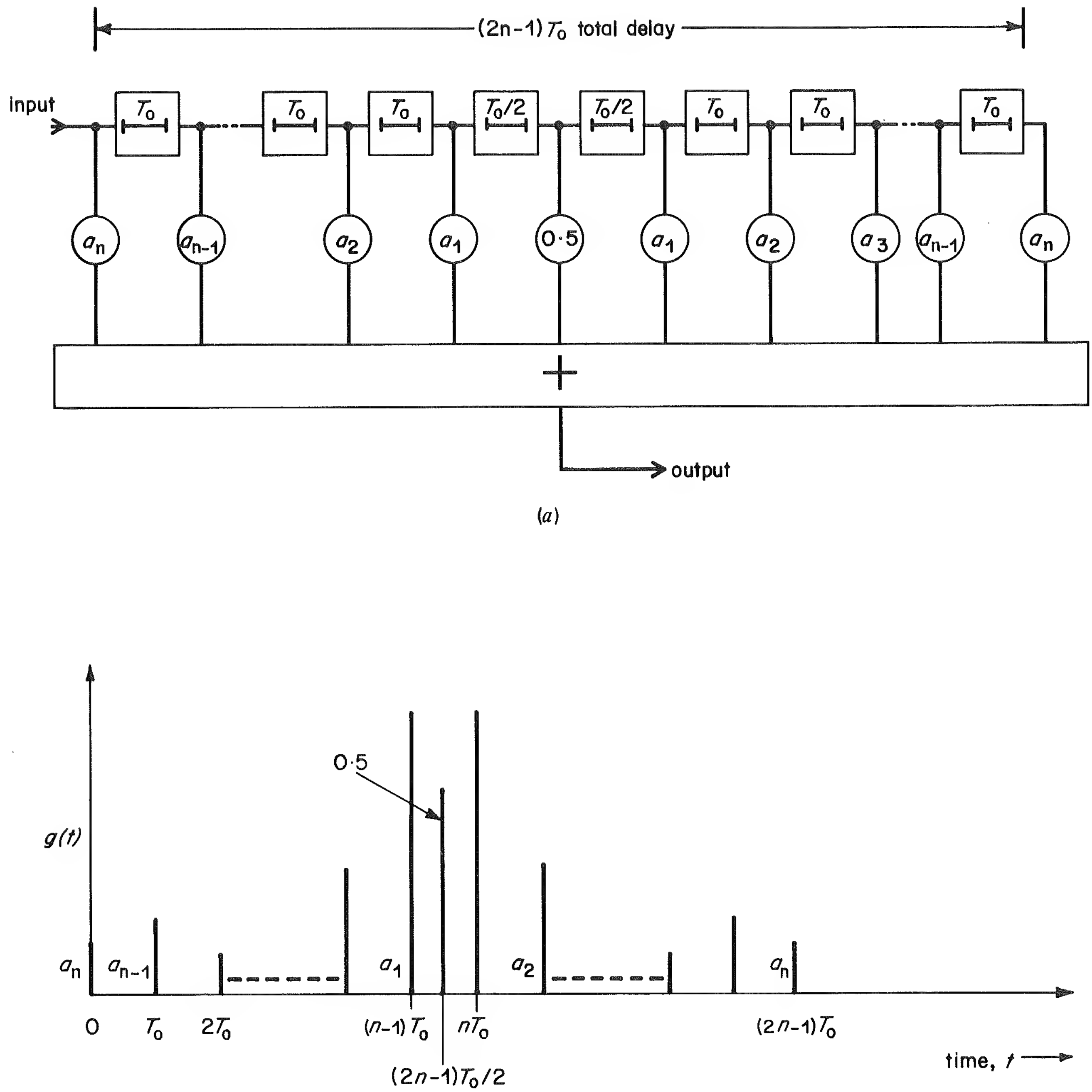
Use of the digital recorder for this experiment has the following advantages and disadvantages:

(i) No analogue imperfections are introduced by the multiple recording process. The filters, a.d.c. and d.a.c. used are all part of the processing experiment.

(ii) Only one machine and one codec are required for multiple recording.

(iii) Any impairments introduced by the recording process take the form of digital errors which are easily disregarded since their visible appearance is quite distinct from the imperfections we are studying (e.g. resolution loss).

(iv) The digital recorder only has the capacity to deal with 7-bit p.c.m. samples. Ideally we should study the cascading of 8-bit sub-Nyquist codecs; the restriction to 7 bits emphasises the effects of quantising, viz. contouring and quantisation noise.



(b)

Fig. 13 - A transversal filter having antisymmetric frequency response
(*a*) Block diagram (*b*) Impulse response

4.3. The filters used for $G(f)$

(a) Two comb filters

The pair of post-coding and pre-coding filters which were used were of the form described in Ref. 3 and were the ones used in stages one and three. Note that the total expression for $G(f)$ with this combination must include the effect of the 5·5 MHz low-pass filters included in the a.d.c. and d.a.c.

(b) Antisymmetric low-pass filter

Appendix 7.5 demonstrates that any transversal filter of the form shown in Fig. 13 must have a frequency response which has the antisymmetric form required to satisfy Equation (11). This means that it is easy to design a filter having such a response using this configuration; an LCR filter would be more difficult to design.

The overall filter transfer function $G(f)$ referred to in

the analysis of Sections 3.2 to 3.4 must of course include the effect of the 5·5 MHz low-pass filters in the a.d.c. and d.a.c., also any filters which might be present in an analogue link. We must therefore choose the coefficients a_r of the transversal filter so that its response above 5·5 MHz is negligible — its response will then dominate so that the overall response satisfies the antisymmetry condition within a sufficient accuracy. Note that ripples in the range 3·37 to 5·5 MHz are entirely acceptable because they will automatically satisfy the antisymmetry condition. However, any ripples in the range 0 to 3·37 MHz will appear in the final insertion response since their 'mirror counterparts' lie above 5·5 MHz and are removed by the extra low-pass filters.

It was decided to make a transversal filter to test the ideas in practice. To keep it very simple, a target specification appropriate to only one passage of the signal through it was chosen, namely that the ripple in the range 0 to 3·37 MHz was restricted to ±1 dB. Calculations showed that this could be achieved by a filter having a total of 5 taps, i.e. 2 coefficient values a_1 and a_2 in addition to the pre-determined value of ½ for the centre-tap coefficient. In fact, a_1 and a_2 were allowed a range of adjustment so that some compensation could be made for the shortcomings of the delay elements used by departing slightly from the ideal settings.

4.4. Experimental results

Multiple recordings were made using several slides and one programme sequence. In every case the conditions of 1, 2, 3, 4 and 5 sub-Nyquist coding operations were recorded. Two sets of recordings were made: one used the comb filter pair in the intermediate re-coding stages, the other used the transversal low-pass filter.

No build-up of diagonal resolution loss or vertical colour resolution loss was observed, as we have predicted. However, some imperfections did build up as follows:

(i) It was difficult to ensure that the gain of analogue parts of the system (between d.a.c. and a.d.c.) was exactly unity, so a slight change of gain and 'sit' was observed.

(ii) A gradual build up of quantising noise and contouring occurred which was more noticeable on some slides than others. This was to be expected and was a result of being limited to 7-bit accuracy by the digital recorder.

(iii) Using the comb filter pair in the intermediate stages a very slight loss at low frequencies occurred which caused visible streaking following vertical luminance edges. Section 3.5 shows that the response $Y(f)$ at *low frequencies* follows $G(f)$. From this we may deduce that the observed l.f. loss in the overall response was merely due to a slight droop in the comb filter response at l.f. which was magnified in effect by the multiple passage of the signal through it. Note that the last stage reached, i.e. 5 codings and decodings, involved 5 passages of the signal through the comb filter pair so that effectively a total of 10 comb filters was involved.

(iv) Using the transversal filter in the intermediate stages produced two defects. One was 'ringing' and emphasis of vertical edges i.e. ripple in the frequency response. This was not surprising — the original filter was designed so that even in the ideal case it had ±1 dB ripple in the range 0 to 3·37 MHz. The practical filter was slightly worse owing to imperfections in the delay sections. Thus, when the signal passed through this filter several times (4 times for 5 codings) this ripple was quite visible.

The second defect was relative delay between chrominance and luminance i.e. difference in group delay for low and high frequencies. This is also caused by the delay elements which exhibited some dispersion.

In summary, apart from quantising effects which are inevitable in any multiple digital coding scheme the only degradations introduced are caused by imperfections in analogue elements — filters, etc. — and were of the form which would equally be caused in analogue systems containing such elements. In particular, the sub-Nyquist nature of the digital coding used did not introduce any special restrictions on the use of multiple coding.

5. Conclusions

It is possible to cascade sampling systems without additive degradation due to sampling provided that certain conditions are met. In particular, the sub-Nyquist sampling of composite PAL video signals may be repeated indefinitely without any build-up of fundamental degradations introduced by the comb filters used in this technique.

The possible future use of mixed analogue and digital transmission does not present a serious obstacle to the use of the sub-Nyquist technique for PAL video signals. An alternative filter for use at intermediate coding points is specified which is somewhat simpler than a comb filter and may permit a less stringent tolerance of the performance of intermediate analogue links.

6. References

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7. APPENDIX

7.1. Allowance for delayed sampling

Consider a sampler which samples an input signal $x(t)$ to produce a sampled output $x_s(t)$, Fig. 14(*a*). If the sampling occurs at instants $t = 0 + nT_o$, classical analysis gives the well-known result for the output spectrum:

$$X_s(f) = \sum_{n=-\infty}^{\infty} X(f - nf_o), \text{ normalised gain being assumed.}$$

What if the sampling were performed at instants $t = \tau + nT_o$? In other words, sampling occurs τ later than in the case first analysed. Conceptually, the same sample values could be obtained if the signal were made to occur τ earlier, sampled at $t = 0 + nT_o$ and then delayed by τ again.

Thus if $x'(t)$ represents the negatively delayed version of $x(t)$ we may write

$$X'(f) = e^{+j\omega\tau} X(f).$$

If $x'(t)$ is sampled, at instants $t = 0 + nT_o$, to give $x'_s(t)$ we obtain:

$$X'_s(f) = \sum_{n=-\infty}^{\infty} X'(f - nf_o)$$

$$= \sum_{n=-\infty}^{\infty} \left\{ X(f - nf_o) e^{j(\omega - n\omega_o)\tau} \right\}$$

If this sampled signal $x'_s(t)$ is delayed by τ we obtain the final output, $x''_s(t)$, which is equivalent to the signal obtained by sampling $x(t)$ at times $t = \tau + nT_o$ (see Figs. 14(*b*) and (*c*)). Thus

$$X''_s(f) = e^{-j\omega\tau} \sum_{n=-\infty}^{\infty} \left\{ X(f - nf_o) e^{j(\omega - n\omega_o)\tau} \right\}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ X(f - nf_o) e^{-jn\omega_o\tau} \right\} \qquad \text{(A1)}$$

7.2. Analysis of repeated-sampling system

We will use the result of 7.1 to continue the analysis started in Section 3.2. The system under consideration is shown in Fig. 4(*b*). Use Equation (A1) to describe the action of the second sampler:

$$S_4(f) = \sum_{n=-\infty}^{\infty} \left\{ S_3(f - nf_o)\, e^{-jn\omega_o\tau} \right\} \qquad \text{(A2)}$$

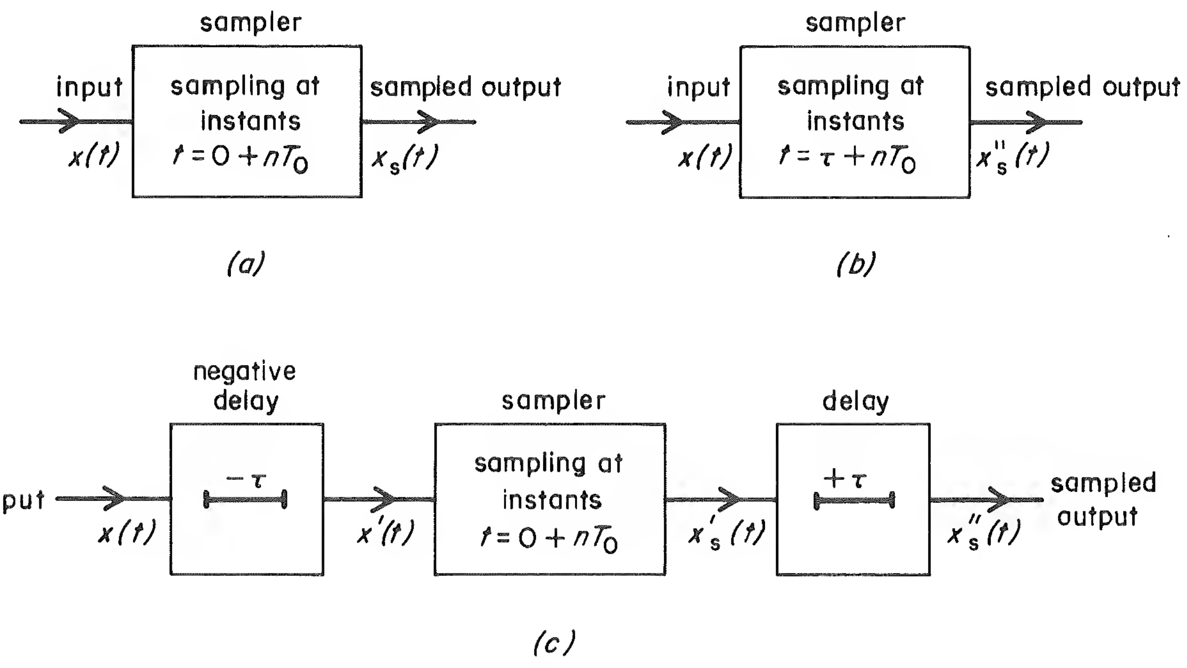

Fig. 14 - Analysis of delayed sampling

(*a*) Simple sampler (*b*) Delayed sampler (*c*) Hypothetical version of (*b*) incorporating (*a*)

Noting that $S_3(f) = G(f).S_2(f)$ and remembering our assumptions regarding low-pass filtering we may substitute from (3) of 3.2 into (A2) to obtain:

$$S_4(f) = G(f)\{S_1(f) + S_1(f+f_o) + S_1(f-f_o)\} +$$
$$+ e^{-j\omega_o\tau}.G(f-f_o)\{S_1(f-f_o) + S_1(f) + S_1(f-2f_o)\}$$
$$+ e^{+j\omega_o\tau}.G(f+f_o)\{S_1(f+f_o) + S_1(f+2f_o) + S_1(f)\} \qquad \text{(A3)}$$

We may simplify this further using our assumptions that

$$\left.\begin{aligned} S_1(f) &= 0 \\ G(f) &= 0 \end{aligned}\right\} \text{ for } |f| \geqslant f_o$$

Considering f positive, in the range $0 \leqslant f < f_o$:

$$S_4(f) = G(f)\{S_1(f) + S_1(f-f_o)\} +$$
$$+ e^{-j\omega_o\tau}\, G(f-f_o)\{S_1(f-f_o) + S_1(f)\}$$
$$= \{G(f) + e^{-j\omega_o\tau}\, G(f-f_o)\} . \{S_1(f) + S_1(f-f_o)\} \qquad \text{(A4)}$$

and for f negative, $0 > f > -f_o$:

$$S_4(f) = G(f)\{S_1(f) + S_1(f+f_o)\} +$$
$$+ e^{+j\omega_o\tau}.G(f+f_o)\{S_1(f+f_o) + S_1(f)\}$$
$$= \{G(f) + e^{+j\omega_o\tau}.G(f+f_o)\}\{S_1(f) + S_1(f+f_o)\} \qquad \text{(A5)}$$

7.3. Derivation and interpretation of $Y(f)$

The effective insertion transfer function, $Y(f)$, was defined by Equation (4) as

$$Y(f) = \frac{S_4(f)}{S_2(f)}.$$

Equation (3) relates $S_2(f)$ to $S_1(f)$ and Equations (A4), (A5) above relate $S_4(f)$ to $S_1(f)$. We may therefore obtain $Y(f)$ by dividing Equation (A4) or (A5) by Equation (3), yielding the results:

for $f_o > f \geqslant 0$: $Y(f) = G(f) + e^{-j\omega_o\tau}\, G(f-f_o)$ (A6)

and for $0 > f > -f_o$: $Y(f) = G(f) + e^{+j\omega_o\tau}\, G(f+f_o)$ (A7)

We will now show that $Y(f)$ behaves like a real base-band filter. Such a filter has a real impulse response $y(t)$. If $y(t)$ is real it follows directly from the Fourier transform definition that

$Y(f) = Y^*(-f)$ and vice versa. (Y^* denotes the complex conjugate of Y).

We will demonstrate that this is true for the present analysis.

Consider the case of positive f and writing $f = -\nu$, so that ν is negative. Then

$Y(-f) = Y(\nu)$

$= G(\nu) + e^{+j\omega_o\tau}G(\nu + f_o)$ from (A7) (valid since ν is negative)

$Y^*(-f) = Y^*(\nu)$

$= G^*(\nu) + e^{-j\omega_o\tau}G^*(\nu + f_o)$ taking complex conjugate of above

$= G(-\nu) + e^{-j\omega_o\tau}G(-\nu - f_o)$ since $G(f)$ describes a real filter

$= G(f) + e^{-j\omega_o\tau}G(f - f_o)$ by substitution

$= Y(f)$ by (A6) (valid since f is positive).

This is the desired result so that $y(t)$ must be real and the combined effect of the arbitrary, but real, filter $G(f)$ and the second sampling operation is therefore equivalent to a real filter of response $Y(f)$.

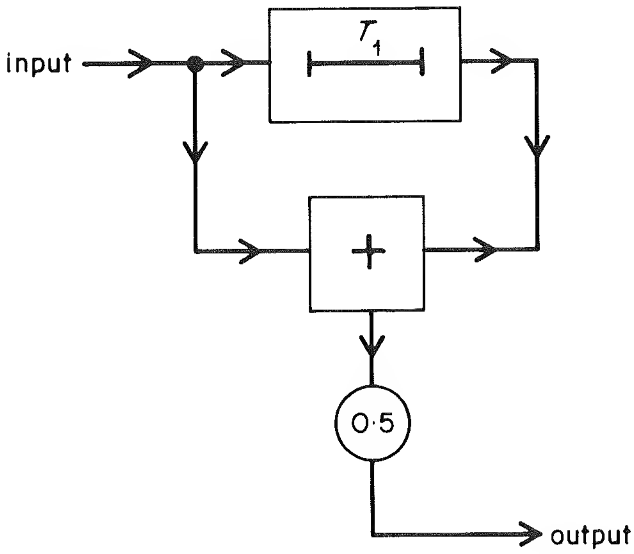

Fig. 15 - Simple comb filter

7.4. Response of simple comb filter

Consider the simple filter of Fig. 15. By inspection, its impulse response, $h_1(t)$ say, is given by

$$h_1(t) = \tfrac{1}{2}\{\delta(t) + \delta(t - T_1)\}$$

Applying the Fourier transform

$$H_1(f) = \tfrac{1}{2}\left(1 + 1.e^{-j\omega T_1}\right)$$

$$= e^{\frac{-j\omega T_1}{2}}\left(\frac{e^{\frac{+j\omega T_1}{2}} + e^{\frac{-j\omega T_1}{2}}}{2}\right)$$

i.e. $$H_1(f) = e^{\frac{-j\omega T_1}{2}} \cos\left(\frac{\omega T_1}{2}\right).$$

If two such filters are connected in cascade the overall response $H_2(f)$, say, is given by

$$H_2(f) = H_1{}^2(f) = e^{-j\omega T_1}\cos^2\left(\frac{\omega T_1}{2}\right)$$

i.e. $$H_2(f) = e^{-j\omega T_1}\left(\frac{1+\cos\omega T_1}{2}\right).$$

Thus $H_2(f)$ is of 'raised cosine' form with constant group delay T_1. Note that for use in PAL sub-Nyquist sampling T_1 would equal 283¾ subcarrier periods so that $T_1 = 567\frac{1}{2}\,T_0$. For convenience we shall write this as $T_1 = (m + \frac{1}{2})T_0$.* The amplitude-frequency response $A(f)$ is the modulus of $H_2(f)$

i.e. $$A(f) = \left(\frac{1+\cos\omega T_1}{2}\right)$$

and $$A(f - f_0) = \frac{\{1 + \cos(\omega - \omega_0)T_1\}}{2}$$

$$= \frac{\{1 + \cos(\omega T_1 - \overline{2m+1}\pi)\}}{2}$$

$$= \frac{1 - \cos\omega T_1}{2}$$

so that

$$A(f) + A(f - f_0) = \frac{(1+\cos\omega T_1)}{2} + \frac{(1-\cos\omega T_1)}{2}$$

$$= 1.$$

Thus Equation (11) of Section 3.3 is satisfied so that a pair of simple comb filters, with the specified delay elements, may be used for $G(f)$ in the arrangement of Fig. 4(b) without any degradation.

7.5. Response of transversal filter of Fig. 13

By inspection of Fig. 13 we can write down the impulse response of the filter as

$$g(t) = a_n\delta(t) + a_{n-1}\,\delta(t - T_0) + \;\ldots\ldots\ldots\ldots$$

$$+ a_1\delta(t - \overline{n-1}T_0) + \tfrac{1}{2}\delta(t - \overline{n-1}T_0 - T_{0/2})$$

$$+ a_1\delta(t - nT_0) + a_2\delta(t - \overline{n+1}T_0) + \;\ldots\ldots\ldots$$

$$+ a_{n-1}\delta(t - \overline{2n-2}T_0) + a_n\delta(t - \overline{2n-1}T_0).$$

i.e. $$g(t) = \tfrac{1}{2}\delta\left(t - \overline{n-1}T_0 - \frac{T_0}{2}\right) + \sum_{r=1}^{n} a_r\left[\delta(t - \overline{n-r}\,T_0) + \delta(t - \overline{n+r-1}T_0)\right].$$

Taking the Fourier transforms of both sides we obtain

$$G(f) = \tfrac{1}{2}.e^{-j\omega(\frac{1}{2} + \overline{n-1})T_0} + \sum_{r=1}^{n} a_r(e^{-j\omega\overline{n-r}T_0} + e^{-j\omega\overline{n+r-1}T_0})$$

$$= e^{-j\omega\left(\frac{2n-1}{2}\right)T_0}\left[\tfrac{1}{2} + \sum_{r=1}^{n} a_r\left\{e^{+j\omega\left(\frac{2r-1}{2}\right)T_0} + e^{-j\omega\left(\frac{2r-1}{2}\right)T_0}\right\}\right]$$

$$= e^{-j\omega\left(\frac{2n-1}{2}\right)T_0}\left[\tfrac{1}{2} + 2\sum_{r=1}^{n} a_r\cos\left(\frac{2r-1}{2}\right)\omega T_0\right].$$

We see that the filter has uniform group delay $(2n-1)/2T_0$ which is half the total of the delay sections used within it. The amplitude frequency response is

$$A(f) = \tfrac{1}{2} + 2\sum_{r=1}^{n} a_r\cos\left(\frac{2r-1}{2}\right)\omega T_0.$$

Consider the rth term, $A_r(f)$ say, where

$$A_r(f) = a_r\cos\left(\frac{2r-1}{2}\right)\omega T_0$$

then $$A_r(f) + A_r(f - f_0) = a_r\left\{\cos\left(\frac{2r-1}{2}\right)\omega T_0 + \cos\left(\frac{2r-1}{2}\right)(\omega - \omega_0)T_0\right\}$$

* Such a relation (with a different value of m) holds for some other sub-Nyquist systems.

$$= 2a_r \cos \frac{(2r-1)(2\omega - \omega_0)T_0}{4} \cdot \cos \frac{(2r-1)\omega_0 T_0}{4}$$

$$= 0 \text{ since } \cos (2r-1)\frac{\pi}{2} = 0.$$

$$\text{Thus } A(f) + A(f-f_0) = ½ + ½ + 2\sum_{r=1}^{n}\left\{A_r(f) + A_r(f-f_0)\right\}$$

$$= 1.$$

The filter therefore satisfies the condition for anti-symmetry of its amplitude-frequency response.

7.6. Derivation of phase response $\phi(f)$ from group delay response $T(f)$

Suppose a network has phase response $\phi(f)$ and group delay response $T(f)$. They are related by

$$T(f) = \frac{-d\phi(f)}{d\omega} = \frac{-1}{2\pi}\frac{d\phi}{df} \qquad \text{(A8)}$$

Integrating, we obtain

$$\phi(f) - \phi(0) = -2\pi \int_0^f T(f)\, df$$

i.e. $\phi(f) = -2\pi \int_0^f T(f)\, df$, since $\phi(0) = 0$ for a real network.

(A9)

Suppose $T(f)$ is specified as a nominal group delay T with variation $\Delta T(f)$, i.e. $T(f) = T + \Delta T(f)$.

Then $$\phi(f) = -2\pi \int_0^f T(f)\, df$$

$$= -\omega T - 2\pi \int_0^f \Delta T(f)\, df \qquad \text{(A10)}$$

We may apply this result to the calculations of Section 3.5. In this case, we wished to know the value of $\theta(f)$, where

$$\theta(f) = \phi'(f-f_0) - \phi'(f)$$

and $$\phi'(f) = \phi(f) + \omega T_c \text{ where } T_c = \frac{-\phi(f_0/2)}{\omega_0/2}$$

Substituting,

$$\theta(f) = \phi(f-f_0) + (\omega - \omega_0)T_c - \phi(f) - \omega T_c$$
$$= \phi(f-f_0) - \phi(f) - \omega_0 T_c$$
$$= \phi(f-f_0) - \phi(f) + 2\phi(f_0/2)$$

Now apply Equation (A10):

$$\theta(f) = -(\omega - \omega_0)T - 2\pi \int_0^{f-f_0} \Delta T(f) df$$

$$+\omega T + 2\pi \int_0^f \Delta T(f) df +$$

[PTO]

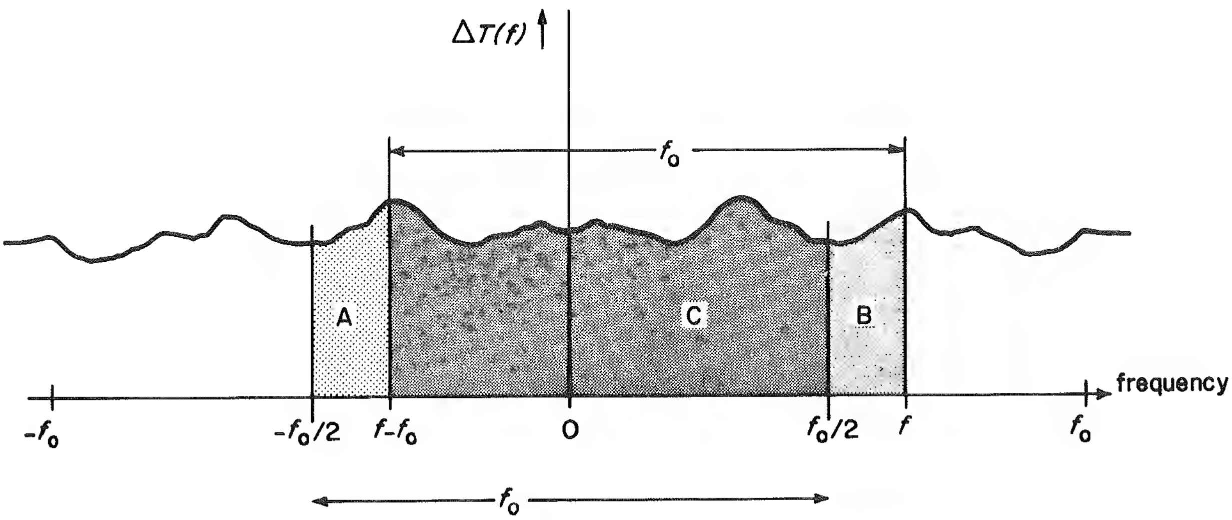

Fig. 16 - To illustrate derivation of phase response from group delay response (see Section 7.6)

$$-2.\omega_o/2.T - 2\pi \int_{o}^{f_{o/2}} \Delta T(f) df$$

$$= -2\pi \left[\int_{o}^{f-f_o} \Delta T(f) df + 2 \int_{o}^{f_{o/2}} \Delta T(f) df - \int_{o}^{f} \Delta T(f) df \right]$$

Noting that $\Delta T(f) = \Delta T(-f)$ for a network with a real impulse response this becomes

$$\theta(f) = -2\pi \left[\int_{-f_{o/2}}^{+f_{o/2}} \Delta T(f) df - \int_{f-f_o}^{f} \Delta T(f) df \right].$$

Refer to Fig. 16. The value of the term within the brackets [ ] is given by

[ ] = (area A + area C) — (area B + area C)

= area A — area B.

Thus $\theta(f) = 2\pi$ (area B — area A). (A11)

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